"APPROXIMATE APPROXIMATIONS" AND THE CUBATURE OF POTENTIALS

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ABSTRACT. The paper discusses new cubature formulas for classical integral operators of mathematical physics based on the "approximate approximation" of the density with Gaussian and related functions. We derive formulas for the cubature of harmonic, elastic and diffraction potentials approximating with high order in some range relevant for numerical computations. We prove error estimates and provide numerical results for the Newton potential.

1. INTRODUCTION

The paper is devoted to the foundation of new cubature formulas for certain integral operators of mathematical physics. It is well known that the numerical treatment of potentials and other integral operators with singularities is an essential part of different methods for the solution of many practical problems. However, the cubature of such integrals is usually very time-consuming, especially in the multidimensional case, such that it is of great importance to derive effective approximations. In this paper we propose formulas which are very simple and provide in numerical computations very accurate approximations. They are based on "approximate approximations", recently proposed by the first author (see Maz'ya [4] and the survey paper Maz'ya [5]). This approximation procedure is mainly directed to the numerical solution of partial integro-differential equations. The method provides simple formulas for quasi–interpolants, which approximate functions up to a prescribed precision very accurately, but in general the approximants do not converge. The lack of convergence, which is not perceptible in numerical computations, is offset by a greater flexibility in the choice of approximating functions. So it is possible to construct multivariate approximation formulas, which are easy to implement and have additionally the property that pseudodifferential operations can be effectively performed. This allows to create effective numerical algorithms for solving boundary value problems for differential and integral equations.

Here we apply the approximation method to the cubature of harmonic, elastic and diffraction potentials, where the obtained cubature formulas result from the approximate approximation of the densities. Based on error estimates for the quasi–interpolation in Sobolev norms we can show that they keep a high approximation rate in a certain range relevant for numerical computations. And in addition, the smoothing properties of the potentials ensure that the cubature formulas converge. For example, the harmonic potential

$$\mathcal{L}_n u(\mathbf{x}) := \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \int\limits_{\mathbf{R}^n} \frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}}\, d\mathbf{y} , \quad n \geq 3 , \tag{1.1}$$

can be approximated by the cubature formula

$$\mathcal{L}_{n,h} u(\mathbf{x}) :=$$

$$\frac{\mathcal{D}h^2}{4(\pi\mathcal{D})^{n/2}} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m}) \left(\frac{1}{|\mathbf{r_m}|^{n-2}} \int\limits_0^{|\mathbf{r_m}|^2} t^{n/2-2} e^{-t}\, dt + e^{-|\mathbf{r_m}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(n/2-1)}(|\mathbf{r_m}|^2)}{j+1} \right), \tag{1.2}$$

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where $\mathcal{D} > 0$ is a fixed parameter, M is an arbitrary integer ≥ 2,

$$\mathbf{r_m} = \frac{\mathrm{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h} \quad \text{and} \quad L_j^{(\alpha)}(y) = \frac{1}{j!}\, e^y y^{-\alpha} \Big(\frac{d}{dy}\Big)^j \left(e^{-y} y^{j+\alpha}\right)$$

are the generalized Laguerre polynomials. The following estimate of the cubature error holds.

Let $1 < p < n/2$, $q = np/(n-2p)$ and suppose that $u \in W_p^{2M+2}(\mathbf{R^n})$, $M > n/2q$. There exist positive constants not depending on u, h and $\mathcal{D}$ such that

$$\|\mathcal{L}_n u - \mathcal{L}_{n,h} u\|_{L_q(\mathbf{R^n})} \leq \left(c_1\, (\sqrt{\mathcal{D}}h)^{2M} + c_2\, h^2\, e^{-\pi^2 \mathcal{D}}\right) \|u\|_{W_p^{2M+2}(\mathbf{R^n})} \,. \tag{1.3}$$

Thus, in general $\mathcal{L}_{n,h}$ gives a second order approximation of $\mathcal{L}_n$. But if the parameter $\mathcal{D}$ is appropriately chosen, then in view of $\exp(-\pi^2) = 0.51723\ldots \cdot 10^{-4}$ the cubature $\mathcal{L}_{n,h}$ behaves in numerical computations like an approximation of the order $2M$.

Let us note that (1.2) is a smooth approximation of $\mathcal{L}_n u$ and that an estimate similar to (1.3) holds also for the gradient of $\mathcal{L}_n u - \mathcal{L}_{n,h} u$ (see Thm. 3.2).

The underlying idea in obtaining cubature formulas of the form (1.2) consists in the approximation of the density u by certain quasi–interpolants

$$u_h(\mathrm{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbf{Z}^n} u(h\mathbf{m})\, \eta\Big(\frac{\mathrm{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) \tag{1.4}$$

with fixed $\mathcal{D} > 0$ and the generating function η is chosen such that the corresponding potential with density η can be effectively determined. Then the cubature formula for the integral operator is obtained by replacing the density u by its quasi–interpolant u_h. The cubature error can be estimated by using mapping properties of the corresponding integral operator and error estimations of the quasi–interpolation (1.4).

The outline of the paper is as follows. In Section 2 we estimate the approximation error $u - u_h$ of the density in L_p and in weaker norms for general η satisfying certain decay and moment conditions. In particular we derive the remarkable fact that approximate approximations, which in general do not converge in L_p, become converging in weaker norms. In Section 3 we describe in detail for the example of the of harmonic potentials how to obtain the cubature formula (1.2) and to prove the estimate (1.3). The last section concerns high order explicit cubature formulas for elastic and diffraction potentials.

2. Quasi–Interpolation Error in L_p and Weak Norms

We consider the quasi–interpolant (1.4) with some generating function η satisfying the decay condition

$$|\eta(\mathbf{t})| \leq A\,(1 + |\mathbf{t}|)^{-K-n-\delta}\,, \quad \mathbf{t} \in \mathbf{R^n}\,, \tag{2.1}$$

for some natural number K and positive constants A and δ, and the moment conditions

$$\int\limits_{\mathbf{R}^n} \eta(\mathbf{t})\, d\mathbf{t} = 1\,, \int\limits_{\mathbf{R}^n} \mathbf{t}^{\boldsymbol{\alpha}} \eta(\mathbf{t})\, d\mathbf{t} = 0, \quad \text{for all } \boldsymbol{\alpha}\,,\ 1 \leq |\boldsymbol{\alpha}| < N < K\,. \tag{2.2}$$

Here and henceforth we use the notations:
Let $\mathrm{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{\geq 0}$ be a multiindex. We denote $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, $\mathrm{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!$,

$$\partial^{\boldsymbol{\alpha}} u(\mathrm{x}) = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u(\mathrm{x})\,.$$

The usual scalar product in $\mathbf{R}^n$ is denoted by $\langle \mathrm{x}, \mathrm{y} \rangle$ and $|\mathrm{x}| = \langle \mathrm{x}, \mathrm{x} \rangle^{1/2}$. With the abbreviation

$$e_{\boldsymbol{\lambda}}(\mathrm{x}) := e^{2\pi i \langle \mathrm{x}, \boldsymbol{\lambda} \rangle}$$

the Fourier transform of an L_1-function is defined by

$$\mathcal{F}\varphi(\boldsymbol{\lambda}) = \int\limits_{\mathbf{R}^n} \varphi(\mathrm{x})\, e_{\boldsymbol{\lambda}}(-\mathrm{x})\, d\mathrm{x}\,.$$

In the following we will use an expansion of the error $u - u_h$, which follows immediately form the Taylor expansion of the function u, the Poisson summation formula (see Stein-Weiss [10]) and the moment conditions (2.2). Let u be a sufficiently smooth function and denote

$$U_{\boldsymbol{\alpha}}(\mathrm{x},\mathrm{y}) := |\boldsymbol{\alpha}| \int\limits_0^1 s^{|\boldsymbol{\alpha}|-1} \partial^{\boldsymbol{\alpha}} u(s\mathrm{x} + (1-s)\mathrm{y})\, ds\,. \tag{2.3}$$

Lemma 2.1. *Let η satisfy* (2.1) *and* (2.2)*. Suppose that for given $\mathcal{D} > 0$ the Fourier transform of η is such that*

$$\{\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\ \cdot)\} \in l_1(\mathbf{Z}^n)\,, \quad 0 \le |\boldsymbol{\alpha}| < N\,. \tag{2.4}$$

Then for any $u \in C_0^\infty(\mathbf{R}^n)$ and $L \le K$ there holds a.e.

$$u_h(\mathrm{x}) - u(\mathrm{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi i}\Big)^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}} u(\mathrm{x})}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} \partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})\, e_{\boldsymbol{\nu}}(h^{-1}\mathrm{x})$$

$$+ \sum_{|\boldsymbol{\alpha}|=N}^{L-1} (-\sqrt{\mathcal{D}}h)^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}} u(\mathrm{x})}{\boldsymbol{\alpha}!} \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big(\frac{\mathrm{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathrm{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) + R_{L,h}(\mathrm{x})\,,$$

where

$$R_{L,h}(\mathrm{x}) = \sum_{|\boldsymbol{\alpha}|=L} \frac{\mathcal{D}^{-n/2}}{\boldsymbol{\alpha}!} \sum_{\mathbf{m}\in\mathbf{Z}^n} \big(h\mathbf{m} - \mathrm{x}\big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathrm{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) U_{\boldsymbol{\alpha}}(\mathrm{x}, h\mathbf{m})\,. \tag{2.5}$$

This expansion is obtained for continuous generating functions in [8], a more special result is also given in Beatson-Light [3]. Next we estimate the norm of $R_{L,h}(\mathrm{x})$ in the function space $L_p = L_p(\mathbf{R}^n)$.

Lemma 2.2. *Suppose that the function u is such that $\partial^{\boldsymbol{\alpha}} u \in L_p$ for all multiindeces $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = L$, where $1 \le p \le \infty$ and $n/p < L \le K$. Then the function $R_{L,h}$ defined in* (2.5) *admits the estimate*

$$\|R_{L,h}\|_{L_p} \le c_\eta\, (\sqrt{\mathcal{D}}h)^L \sum_{|\boldsymbol{\alpha}|=L} \frac{\|\partial^{\boldsymbol{\alpha}} u\|_{L_p}}{\boldsymbol{\alpha}!}$$

with some constant c_η not depending on u, h and $\mathcal{D}$.

Proof. Let us define the functions

$$\phi_j(s) := \max_{\mathrm{x}\in\mathbf{R}^n} s^{-n} \sum_{\mathbf{m}\in\mathbf{Z}^n} \big(1 + |s^{-1}\mathbf{m} - \mathrm{x}|\big)^{-j-n-\delta}\,, \quad j \ge 0\,,$$

which continuously depend on $s \in (0,\infty)$ and for $s \to \infty$ we have

$$\phi_j(s) \ \to \int\limits_{\mathbf{R}^n} (1 + |\mathrm{x}|)^{-j-n-\delta}\, d\mathrm{x} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{\Gamma(n)\Gamma(j+\delta)}{\Gamma(n+j+\delta)}\,.$$

Hence for any $t \in (0,\infty)$ there exist constants $\Phi_j(t)$ such that

$$\phi_j(s) \le \Phi_j(t)\,, \quad \text{if } \ t \le s < \infty\,. \tag{2.6}$$

Further, the functions $s^n \phi_j(s)$ are increasing, thus

$$\phi_j(s) \le (t/s)^n\, \Phi_j(t)\,, \quad \text{for } \ 0 < s \le t\,. \tag{2.7}$$

Note that the decay condition (2.1) implies in particular

$$\Big\|(\sqrt{\mathcal{D}}s)^{-n}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big|\Big(\frac{\cdot-\mathbf{m}}{\sqrt{\mathcal{D}}s}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\cdot-\mathbf{m}}{\sqrt{\mathcal{D}}s}\Big)\Big|\Big\|_{L_\infty}\le A\,s^{-n}\,\Phi_{K-L}(\sqrt{\mathcal{D}})\,. \tag{2.8}$$

for any $s\in(0,1]$ and $|\boldsymbol{\alpha}|=L$.

To estimate $\|R_{L,h}\|_{L_p}$ we introduce the functions

$$S_{\boldsymbol{\alpha},h}(\mathbf{x}):=\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)U_{\boldsymbol{\alpha}}(\mathbf{x},h\mathbf{m})\,,$$

such that

$$\|R_{L,h}\|_{L_p}\le(\sqrt{\mathcal{D}}h)^L\sum_{|\boldsymbol{\alpha}|=L}\frac{1}{\boldsymbol{\alpha}!}\,\|S_{\boldsymbol{\alpha},h}\|_{L_p}\,.$$

If $1<p<\infty$ we apply Hölder's inequality to

$$\begin{aligned}\|S_{\boldsymbol{\alpha},h}\|_{L_p}^p&\le\int_{\mathbf{R}^n}\Big(\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big|\mathcal{D}^{-n/2}\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\Big|^{1/p+1/p'}|U_{\boldsymbol{\alpha}}(\mathbf{x},h\mathbf{m})|\Big)^p d\mathbf{x}\\&\le\big(A\Phi_{K-L}(\sqrt{\mathcal{D}})\big)^{p/p'}\int_{\mathbf{R}^n}\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big|\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\Big|\,|U_{\boldsymbol{\alpha}}(\mathbf{x},h\mathbf{m})|^p d\mathbf{x}\,,\end{aligned}$$

where $1/p+1/p'=1$ and we used (2.8) with $s=1$. We choose a number θ satisfying $0<\theta<L-n/p$ and apply once more the Hölder inequality to get

$$\begin{aligned}|U_{\boldsymbol{\alpha}}(\mathbf{x},h\mathbf{m})|^p&=L^p\,\Big|\int_0^1 s^{L-1}\partial^{\boldsymbol{\alpha}}u(s\mathbf{x}+(1-s)h\mathbf{m})ds\Big|^p\\&\le L^p\int_0^1 s^{(L-1-1/p'-\theta)p}|\partial^{\boldsymbol{\alpha}}u(s\mathbf{x}+(1-s)h\mathbf{m})|^p ds\,\Big\{\int_0^1 s^{-1+\theta p'}ds\Big\}^{p/p'}\\&=\frac{L^p}{(\theta p')^{p/p'}}\int_0^1 s^{(L-\theta)p-1}|\partial^{\boldsymbol{\alpha}}u(s\mathbf{x}+(1-s)h\mathbf{m})|^p ds\,.\end{aligned}$$

Hence it remains to estimate the expression

$$\begin{aligned}&\int_{\mathbf{R}^n}\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big|\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\Big|\int_0^1 s^{(L-\theta)p-1}|\partial^{\boldsymbol{\alpha}}u(s\mathbf{x}+(1-s)h\mathbf{m})|^p ds\,d\mathbf{x}\\&=\int_0^1 s^{(L-\theta)p-1}\int_{\mathbf{R}^n}|\partial^{\boldsymbol{\alpha}}u(\mathbf{t})|^p\,(\sqrt{\mathcal{D}}s)^{-n}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big|\Big(\frac{\mathbf{t}-h\mathbf{m}}{\sqrt{\mathcal{D}}hs}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\mathbf{t}-h\mathbf{m}}{\sqrt{\mathcal{D}}hs}\Big)\Big|d\mathbf{t}\,ds\,,\end{aligned}$$

where we changed several times the order of integrations and summation, which is justified since the integrands are nonnegative, and substituted $\mathbf{t}=s\mathbf{x}+(1-s)h\mathbf{m}$. We apply (2.8) to obtain

$$\begin{aligned}&\int_0^1 s^{(L-\theta)p-1}\int_{\mathbf{R}^n}|\partial^{\boldsymbol{\alpha}}u(\mathbf{t})|^p\,(\sqrt{\mathcal{D}}s)^{-n}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big|\Big(\frac{\mathbf{t}-h\mathbf{m}}{\sqrt{\mathcal{D}}hs}\Big)^{\boldsymbol{\alpha}}\eta\Big(\frac{\mathbf{t}-h\mathbf{m}}{\sqrt{\mathcal{D}}hs}\Big)\Big|d\mathbf{t}\,ds\\&\le A\,\Phi_{K-L}(\sqrt{\mathcal{D}})\int_0^1 s^{(L-\theta)p-1-n}ds\int_{\mathbf{R}^n}|\partial^{\boldsymbol{\alpha}}u(\mathbf{t})|^p d\mathbf{t}\;=\;\frac{A\,\Phi_{K-L}(\sqrt{\mathcal{D}})}{(L-\theta)p-n}\,\|\partial^{\boldsymbol{\alpha}}u\|_{L_p}^p\,,\end{aligned}$$

leading to

$$\|S_{\boldsymbol{\alpha},h}\|_{L_p}^p \le \left(A\Phi_{K-L}(\sqrt{\mathcal{D}})\right)^{p/p'} \frac{L^p}{(\theta p')^{p/p'}} \frac{A\,\Phi_{K-L}(\sqrt{\mathcal{D}})}{(L-\theta)p-n} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p}^p$$

for arbitrary $\theta \in (0, L-n/p)$. But

$$\min_{0<\theta<L-n/p} \frac{1}{(\theta p')^{1/p'}((L-\theta)p-n)^{1/p}} = \frac{p}{Lp-n} \ ,$$

therefore we obtain

$$\|S_{\boldsymbol{\alpha},h}\|_{L_p} \le A\,\Phi_{K-L}(\sqrt{\mathcal{D}}) \frac{Lp}{Lp-n} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p} \ .$$

Using (2.8) with $s=1$ and the inequality $|U_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y})| \le \|\partial^{\boldsymbol{\alpha}} u\|_{L_\infty}$ we obtain for $p=\infty$ that

$$\|S_{\boldsymbol{\alpha},h}\|_{L_\infty} \le A\Phi_{K-L}(\sqrt{\mathcal{D}})\|\partial^{\boldsymbol{\alpha}} u\|_{L_\infty} \ .$$

If $p=1$ then clearly

$$\|S_{\boldsymbol{\alpha},h}\|_{L_1} \le L\int_0^1 s^{L-1} \int_{\mathbf{R}^n} |\partial^{\boldsymbol{\alpha}} u(\mathbf{t})|(\sqrt{\mathcal{D}}s)^{-n} \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big|\Big(\frac{\mathbf{t}-h\mathbf{m}}{\sqrt{\mathcal{D}}hs}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathbf{t}-h\mathbf{m}}{\sqrt{\mathcal{D}}hs}\Big)\Big|\, d\mathbf{t}\, ds$$

$$\le A\,L\,\Phi_{K-L}(\sqrt{\mathcal{D}}) \int_0^1 s^{L-1-n} ds \int_{\mathbf{R}^n} |\partial^{\boldsymbol{\alpha}} u(\mathbf{t})| d\mathbf{t} \ \le \ \frac{A\,L\,\Phi_{K-L}(\sqrt{\mathcal{D}})}{L-n} \|\partial^{\boldsymbol{\alpha}} u\|_{L_1} \ .$$

Thus the assertion is proved for $1 \le p \le \infty$ and we see that the constant c_η is bounded by

$$c_\eta \ \le \ A\,\Phi_{K-L}(\sqrt{\mathcal{D}}) \frac{Lp}{Lp-n} \ .$$

∎

For the following we introduce the semi–norm

$$|u|_{W_p^k} := \sum_{|\boldsymbol{\alpha}|=k} \frac{k!}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p} \ ,$$

equip the Sobolev space $W_p^l = W_p^l(\mathbf{R}^n)$, $l \in \mathbf{N}$, with the norm $\|u\|_{W_p^l} = \sum_{k=0}^{l} |u|_{W_p^k}$ and denote

$$\begin{aligned} \sigma_k(\eta,\mathcal{D}) &:= \max_{|\boldsymbol{\alpha}|=k} \Big\| \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big(\frac{\cdot-\mathbf{m}}{\sqrt{\mathcal{D}}}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\cdot-\mathbf{m}}{\sqrt{\mathcal{D}}}\Big)\Big\|_{L_\infty} \ , \\ \varepsilon_k(\eta,\mathcal{D}) &:= \max_{|\boldsymbol{\alpha}|=k} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})| \end{aligned} \tag{2.9}$$

Due to (2.6) there holds $\sigma_k(\eta,\mathcal{D}) \ \le \ A\,\Phi_{K-k}(\sqrt{\mathcal{D}})$. Now Lemmas 2.1 and 2.2 lead to the following L_p–estimate of approximate approximations.

Theorem 2.1. *Suppose that η satisfies* (2.1) *and* (2.2) *and that for given $\mathcal{D}>0$ the relations* (2.4) *hold. Then for any $h>0$ and any function $u \in W_p^L$, $1 \le p \le \infty$, $L \ge N$ and $n/p < L \le K$ we have*

$$\|u_h - u\|_{L_p} \le \sum_{k=0}^{N-1} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^k \frac{\varepsilon_k(\eta,\mathcal{D})}{k!} |u|_{W_p^k}$$
$$+ \sum_{k=N}^{L-1} \frac{(\sqrt{\mathcal{D}}h)^k \sigma_k(\eta,\mathcal{D})}{k!} |u|_{W_p^k} + \frac{c_\eta\,(\sqrt{\mathcal{D}}h)^L}{L!} |u|_{W_p^L} \ .$$

Now we are going to estimate the error of $u-u_h$ in negative norms.

Lemma 2.3. *Let $u \in W_p^{2r}$, $1 \le p < \infty$, $r \in \mathbf{N}$, and $\{a_{\boldsymbol\nu}\} \in l_1(\mathbf{Z}^n)$. Then there exists a constant c depending only on n, r and p such that*

$$\| u \sum_{\boldsymbol\nu \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} a_{\boldsymbol\nu} e_{\boldsymbol\nu}(h^{-1}\cdot)\|_{W_p^{-2r}} \le c\, h^{2r} \sum_{\boldsymbol\nu \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} \frac{|a_{\boldsymbol\nu}|}{(4\pi^2|\boldsymbol\nu|^2)^r} \|u\|_{W_p^{2r}} .$$

Proof. Obviously it suffices to estimate the norm of the multiplier

$$e_{\boldsymbol\nu}(h^{-1}\cdot) : W_p^{2r} \to W_p^{-2r} .$$

Because of $|\boldsymbol\nu| \ne 0$ we get

$$\begin{aligned}
\|e_{\boldsymbol\nu}(h^{-1}\cdot)\, u\|_{W_p^{-2r}} &= \sup_{\|v\|_{W_{p'}^{2r}}=1} \Big| \int\limits_{\mathbf{R}^n} e_{\boldsymbol\nu}(h^{-1}\mathrm{x})\, u(\mathrm{x})\, \overline{v(\mathrm{x})}\, d\mathrm{x} \Big| \\
&= \sup_{\|v\|_{W_{p'}^{2r}}=1} \Big| \frac{h^{2r}}{(4\pi^2|\boldsymbol\nu|^2)^r} \int\limits_{\mathbf{R}^n} (-\Delta)^r e_{\boldsymbol\nu}(h^{-1}\mathrm{x})\, u(\mathrm{x})\, \overline{v(\mathrm{x})}\, d\mathrm{x} \Big| \\
&\le \frac{h^{2r}}{(4\pi^2|\boldsymbol\nu|^2)^r} \sup_{\|v\|_{W_{p'}^{2r}}=1} \int\limits_{\mathbf{R}^n} |\Delta^r (u(\mathrm{x})\, \overline{v(\mathrm{x})})|\, d\mathrm{x} .
\end{aligned}$$

Since

$$\Delta^r(u\,\overline{v}) = \sum_{|\boldsymbol\alpha|=r} \frac{r!}{\boldsymbol\alpha!} \partial^{2\boldsymbol\alpha}(u\,\overline{v}) = \sum_{|\boldsymbol\alpha|=r} \frac{r!}{\boldsymbol\alpha!} \sum_{\boldsymbol\beta \le 2\boldsymbol\alpha} \frac{(2\boldsymbol\alpha)!}{\boldsymbol\beta!\,(2\boldsymbol\alpha-\boldsymbol\beta)!} \partial^{\boldsymbol\beta} u\, \partial^{2\boldsymbol\alpha-\boldsymbol\beta}\overline{v}$$

it is clear that there exists a constant depending only on n, r and p such that

$$\int\limits_{\mathbf{R}^n} |\Delta^r(u(\mathrm{x})\, \overline{v(\mathrm{x})})|\, d\mathrm{x} \le c\, \|u\|_{W_p^{2r}} \|v\|_{W_{p'}^{2r}} .$$

We note that in the special case $r = 1$ this constant is bounded by

$$c \le \max(2, n^{1/\tilde p}) , \quad \tilde p = \min(p, p') .$$

∎

By simple interpolation arguments Lemma 2.3 can be generalized for arbitrary negative norms in Besov and Bessel potential spaces. This leads to the following error estimation for the quasi–interpolation formula (1.4), which we formulate for the example of the Bessel potential space $H_p^s = H_p^s(\mathbf{R}^n)$ equipped with the norm

$$\|u\|_{H_p^s} = \|\mathcal{F}^{-1}(1 + 4\pi^2|\cdot|)^{s/2}\mathcal{F}u\|_{L_p} = \|(I - \Delta)^{s/2}u\|_{L_p} .$$

Theorem 2.2. *Suppose that η satisfies* (2.1), (2.2) *and for given $\mathcal{D} > 0$ the relations* (2.4) *hold. Then for any $u \in H_p^L$, $1 < p < \infty$, $L \ge N$ with $n/p < L \le K$ and positive $s \le L$ there exist constants c_η and $c_{s,p}$ not depending on u and h such that*

$$\begin{aligned}
&\|u - u_h\|_{H_p^{-s}} \le c_\eta\, (\sqrt{\mathcal{D}}h)^N \|u\|_{H_p^L} \\
&+ c_{s,p}\, h^s \sum_{|\boldsymbol\alpha|=0}^{\min(N-1,[L-s])} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^{|\boldsymbol\alpha|} \frac{\|\partial^{\boldsymbol\alpha} u\|_{H_p^s}}{\boldsymbol\alpha!} \sum_{\boldsymbol\nu \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} \frac{|\partial^{\boldsymbol\alpha} \mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol\nu)|}{(2\pi|\boldsymbol\nu|)^s} .
\end{aligned}$$

Proof. Since $\|\cdot\|_{H_p^s} \le \|\cdot\|_{H_p^t}$ for $s < t$ the estimate

$$\begin{aligned}
\|u - u_h\|_{H_p^{-s}} \le & \|R_{L,h}\|_{L_p} + \sum_{k=N}^{L-1} \frac{(\sqrt{\mathcal{D}}h)^k \sigma_k(\eta, \mathcal{D})}{k!} |u|_{W_p^k} \\
& + \sum_{|\boldsymbol\alpha|=0}^{N-1} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^{|\boldsymbol\alpha|} \frac{1}{\boldsymbol\alpha!} \|\, \partial^{\boldsymbol\alpha} u\, \varepsilon_{\boldsymbol\alpha}(h^{-1}\cdot)\, \|_{H_p^{-s}}
\end{aligned}$$

follows from Lemma 2.1 and Theorem 2.1, where we denote

$$\varepsilon_{\boldsymbol{\alpha}}(\mathrm{x}) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} \partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})\, e_{\boldsymbol{\nu}}(\mathrm{x})\,.$$

Obviously the first two summands can be bounded by $c\,(\sqrt{\mathcal{D}}h)^N\,\|u\|_{H_p^L}$ with some constant not depending on u, h and $\mathcal{D}$. To estimate the last sum we use for $|\boldsymbol{\alpha}|+s\le L$ the interpolated result of Lemma 2.3 to get

$$\|\partial^{\boldsymbol{\alpha}}u\,\varepsilon_{\boldsymbol{\alpha}}(h^{-1}\,\cdot\,)\,\|_{H_p^{-s}} \le c_{s,p}\,h^s\,\|\partial^{\boldsymbol{\alpha}}u\|_{H_p^s} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} \frac{|\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})|}{(2\pi|\boldsymbol{\nu}|)^s}\,.$$

If $|\boldsymbol{\alpha}|+s>L$ we use Lemma 2.3 for $u\in H_p^{L-|\boldsymbol{\alpha}|}$ and obtain

$$\|\partial^{\boldsymbol{\alpha}}u\,\varepsilon_{\boldsymbol{\alpha}}(h^{-1}\,\cdot\,)\,\|_{H_p^{-s}} \le c_{L-|\boldsymbol{\alpha}|,p}\,h^{L-|\boldsymbol{\alpha}|}\,\|u\|_{H_p^L} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} \frac{|\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})|}{(2\pi|\boldsymbol{\nu}|)^{L-|\boldsymbol{\alpha}|}}\,,$$

showing that

$$\sum_{|\boldsymbol{\alpha}|>L-s}^{N-1} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^{|\boldsymbol{\alpha}|}\frac{1}{\boldsymbol{\alpha}!}\|\partial^{\boldsymbol{\alpha}}u\,\varepsilon_{\boldsymbol{\alpha}}(h^{-1}\,\cdot\,)\,\|_{H_p^{-s}} \le c\,(\sqrt{\mathcal{D}}h)^N\,\|u\|_{H_p^L}\,.$$

∎

Since $\|u\|_{H_p^{-2}} \le B_{p'}\,\|u\|_{W_p^{-2}}$ with the constant $B_{p'} = \sup\limits_{v\in W_{p'}^2} \dfrac{\|v\|_{W_{p'}^2}}{\|(I-\Delta)v\|_{L_{p'}}}$ we obtain for the special case $s=2$

Proposition 2.1. *Suppose that η satisfies* (2.1), (2.2) *and for given $\mathcal{D}>0$ the relations* (2.4) *hold. Then for any $u\in W_p^L$, $L\ge N\ge 2$ with $n/p<L\le K$ there exists a constant c not depending on u, h and $\mathcal{D}$ such that*

$$\begin{aligned}\|u-u_h\|_{H_p^{-2}} &\le c\,(\sqrt{\mathcal{D}}h)^N\,\|u\|_{W_p^L}\\ &\quad + c_p\,h^2\sum_{|\boldsymbol{\alpha}|=0}^{N-3}\Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^{|\boldsymbol{\alpha}|}\frac{\|\partial^{\boldsymbol{\alpha}}u\|_{W_p^2}}{\boldsymbol{\alpha}!}\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}\frac{|\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})|}{4\pi^2|\boldsymbol{\nu}|^2}\\ &\le\; c\,(\sqrt{\mathcal{D}}h)^N\,\|u\|_{W_p^L} + c_p\,h^2\sum_{k=0}^{N-3}\frac{(\sqrt{\mathcal{D}}h)^k}{(2\pi)^{k+2}}\,\frac{\varepsilon_k(\eta,\mathcal{D})}{k!}\sum_{l=0}^{2}|u|_{W_p^{k+l}}\,,\end{aligned}$$

where $c_p = B_{p'}\,\max(2,n^{1/\tilde{p}})$ and $\varepsilon_k(\eta,\mathcal{D})$ are defined in (2.9).

3. Harmonic Potentials

Here we justify the cubature formula (1.2) for the harmonic potential $\mathcal{L}_n$. It is well–known that $\mathcal{L}_n=(-\Delta)^{-1}$ and that for $n\ge 3$, $1<p<n/2$ and $q=np/(n-2p)$ the operator $\mathcal{L}_n$ is a bounded mapping from L_p into L_q (cf. Stein [9]). The norm of $\mathcal{L}_n : L_p\to L_q$ we denote by $A_{p,q}$

Let us define

$$\mathcal{L}_{n,h}u(\mathrm{x}) := \mathcal{L}_n u_h(\mathrm{x}) = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}}\int\limits_{\mathbf{R}^n}\frac{u_h(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{n-2}}\,d\mathrm{y}\,, \tag{3.1}$$

where u_h is given by (1.4). Hence for any $\mathbf{x} \in \mathbf{R^n}$ we obtain the discrete formula

$$\mathcal{L}_{n,h} u(\mathbf{x}) = \frac{h^2}{\mathcal{D}^{n/2-1}} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\, \mathcal{L}_n \eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) . \tag{3.2}$$

The cubature error can be estimated by the following

Theorem 3.1. *Let $1 < p < n/2$, $q = np/(n-2p)$ and suppose that η satisfies* (2.1), (2.2) *and for given $\mathcal{D} > 0$ the relations* (2.4) *hold. Then for any function $u \in W_p^L$, where $L \geq N+2$ with $n/p < L \leq K$, there exists a constant c_η not depending on u, h and $\mathcal{D}$ such that*

$$\begin{aligned}
\|\mathcal{L}_n u - \mathcal{L}_{n,h} u\|_{L_q} \leq\ & c_\eta\, (\sqrt{\mathcal{D}}h)^N \|u\|_{W_p^L} \\
& + h^2 \sum_{k=0}^{N-3} \frac{(\sqrt{\mathcal{D}}h)^k}{(2\pi)^{k+2}} \frac{\varepsilon_k(\eta,\mathcal{D})}{k!} \sum_{l=0}^{2} (A_{p,q}\, c_p\, |u|_{W_p^{k+l}} + c_q\, |u|_{W_q^{k+l}}) .
\end{aligned}$$

Proof. The assertion follows immediately from Proposition 2.1 and the mapping properties of the operator $\mathcal{L}_n$. Since $\mathcal{L}_n u - \mathcal{L}_{n,h} u = \mathcal{L}_n(u - u_h)$ we obtain

$$\begin{aligned}
&\|\mathcal{L}_n u - \mathcal{L}_{n,h} u\|_{L_q} = \|(-\Delta)^{-1}(I-\Delta)(I-\Delta)^{-1}(u-u_h)\|_{L_q} \\
&\leq \|(-\Delta)^{-1}(I-\Delta)^{-1}(u-u_h)\|_{L_q} + \|(I-\Delta)^{-1}(u-u_h)\|_{L_q} \\
&\leq A_{p,q}\, \|u-u_h\|_{H_p^{-2}} + \|u-u_h\|_{H_q^{-2}} .
\end{aligned}$$

Using the continuous embedding $W_p^L \subset W_q^{L-2}$ we have only to apply the estimate of Proposition 2.1. ∎

Let us mention an interesting feature of the cubature formulas based on the approximate approximation of the density. As an example we consider the approximation of the gradient $\nabla(\mathcal{L}_n u)(\mathbf{x})$ by the discrete formula

$$\nabla(\mathcal{L}_{n,h} u)(\mathbf{x}) = \frac{h}{\mathcal{D}^{(n-1)/2}} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\, \nabla(\mathcal{L}_n \eta)\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) .$$

Theorem 3.2. *Let $1 < p < n$, $q = np/(n-p)$ and suppose that η satisfies* (2.1), (2.2) *and for given $\mathcal{D} > 0$ the relations* (2.4) *hold. Then for any function $u \in W_p^L$, where $L \geq N+1$ with $n/p < L \leq K$, there exist constants c_η and c_p not depending on u, h and $\mathcal{D}$ such that*

$$\begin{aligned}
\|\nabla(\mathcal{L}_n u) - \nabla(\mathcal{L}_{n,h} u)\|_{L_q} \leq\ & c_\eta\, (\sqrt{\mathcal{D}}h)^N \|u\|_{H_p^L} \\
& + c_p\, h \sum_{|\boldsymbol{\alpha}|=0}^{N-2} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^{|\boldsymbol{\alpha}|} \frac{\|\partial^{\boldsymbol{\alpha}} u\|_{H_p^1}}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} \frac{|\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})|}{2\pi|\boldsymbol{\nu}|} .
\end{aligned}$$

Proof. It is well–known that $\|\nabla u\|_{L_q} \asymp \|(-\Delta)^{1/2} u\|_{L_q}$. Acting as in the proof of the previous theorem we get

$$\|(-\Delta)^{1/2}(\mathcal{L}_n u - \mathcal{L}_{n,h} u)\|_{L_q} \leq B_{p,q}\, \|u-u_h\|_{H_p^{-1}} + \|u-u_h\|_{H_q^{-1}} ,$$

where $B_{p,q}$ denotes the norm of the bounded mapping $(-\Delta)^{-1/2} : L_p \to L_q$ (cf. Stein [9]). Hence by Theorem 2.2 the assertion follows immediately. ∎

The previous theorems show, that if the generating function η and the parameter $\mathcal{D}$ are chosen such that the values $\varepsilon_k(\eta,\mathcal{D})$ are sufficiently small, then both the cubature $\mathcal{L}_{n,h} u$ and its gradient $\nabla(\mathcal{L}_{n,h} u)$ approximate with the order h^N up to the prescribed accuracy. Moreover, due to the smoothing properties of the integral operators the corresponding approximations converge with the rate h^2 and h, respectively, as h tends to zero. This property holds, in general, also for other pseudodifferential operators of negative order, whereas for singular integral operators the corresponding cubatures approximate in some range of h with the order N, but do not converge.

After having estimated the cubature error we choose now a generating function η such that the assumptions of Theorem 3.1 are satisfied, the values $\varepsilon_k(\eta, \mathcal{D})$ can be made arbitrarily small by a proper choice of $\mathcal{D}$ and the integral $\mathcal{L}_n\eta(\mathbf{x})$ can be determined effectively.

An example of multivariate functions providing the desired properties is given by

$$\eta_{2M}(\mathbf{x}) := \pi^{-n/2}\, L_{M-1}^{(n/2)}(|\mathbf{x}|^2)\, e^{-|\mathbf{x}|^2}\ , \quad M = 1, 2, \dots\ , \tag{3.3}$$

having the Fourier transform

$$\mathcal{F}\eta_{2M}(\boldsymbol{\lambda}) = e^{-\pi^2|\boldsymbol{\lambda}|^2} \sum_{j=0}^{M-1} \frac{(\pi^2|\boldsymbol{\lambda}|^2)^j}{j!}\ . \tag{3.4}$$

(see [5], [7]). Obviously the function η_{2M} satisfies (2.1) and (2.4) for any $K > 0$ and the moment conditions (2.2) with $N = 2M$. It will be shown that the cubature formula (1.2) is based on this function. Consequently, to prove the validity of the estimate (1.3) it remains to bound the error

$$\sum_{k=0}^{2M-3} \frac{(\sqrt{\mathcal{D}}h)^k}{(2\pi)^{k+2}}\, \frac{\varepsilon_k(\eta_{2M}, \mathcal{D})}{k!} \sum_{l=0}^{2} (A_{p,q}\, c_p\, |u|_{W_p^{k+l}} + c_q\, |u|_{W_q^{k+l}})\ .$$

In view of (3.4) we have

$$\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta_{2M}(\boldsymbol{\lambda}) = p_{2M-2+|\boldsymbol{\alpha}|}(\boldsymbol{\lambda})\, e^{-\pi^2|\boldsymbol{\lambda}|^2}$$

with some polynomial of total degree $2M - 2 + |\boldsymbol{\alpha}|$. Due to the rapid decay of the Gaussian function there exist constants γ_k such that say for all $\mathcal{D} \geq 1$ we have

$$\varepsilon_k(\eta_{2M}, \mathcal{D}) = \max_{|\boldsymbol{\alpha}|=k} \sum_{\boldsymbol{\nu} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta_{2M}(\sqrt{\mathcal{D}}\boldsymbol{\nu})| \leq \gamma_k\, e^{-\pi^2\mathcal{D}}\ ,$$

implying

$$\sum_{k=0}^{2M-3} \frac{(\sqrt{\mathcal{D}}h)^k}{(2\pi)^{k+2}}\, \frac{\varepsilon_k(\eta_{2M}, \mathcal{D})}{k!} \sum_{l=0}^{2} (A_{p,q}\, c_p\, |u|_{W_p^{k+l}} + c_q\, |u|_{W_q^{k+l}}) \leq c_2\, e^{-\pi^2\mathcal{D}}\, \|u\|_{W_p^{2M+1}}$$

for any fixed $\mathcal{D} \geq 1$ and all $h \leq \mathcal{D}^{-1/2}$. Now Theorem 3.1 yields the estimate of the cubature error

$$\|\mathcal{L}_n u - \mathcal{L}_{n,h} u\|_{L_q} \leq c_1\, (\sqrt{\mathcal{D}}h)^{2M} \|u\|_{W_p^{2M+2}} + c_2\, h^2\, e^{-\pi^2\mathcal{D}}\, \|u\|_{W_p^{2M+1}} \tag{3.5}$$

with constants not depending on u, h and $\mathcal{D} \geq 1$, where

$$\mathcal{L}_{n,h} u(\mathbf{x}) = \frac{h^2}{\mathcal{D}^{n/2-1}} \sum_{\mathbf{m} \in \mathbf{Z}^n} u(h\mathbf{m})\, \mathcal{L}_n\eta_{2M}\Big(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\ . \tag{3.6}$$

Remark: The estimate (3.5) indicates that asymptotically the optimal cubature error can be obtained if the parameter $\mathcal{D}$ is coupled to the gridsize h such that $(\sqrt{\mathcal{D}}h)^{2M} = h^2\, \exp(-\pi^2\mathcal{D})$. A similar coupling of $\mathcal{D}$ and h, depending of course on the generating function η, converts the quasi–interpolation (1.4) into a converging process. For example, if for given η we choose $\mathcal{D} = \mathcal{D}(h)$ such that

$$\max_{0 \leq k < N} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^k\, \frac{\varepsilon_k(\eta, \mathcal{D})}{k!} = (\sqrt{\mathcal{D}}h)^N\ ,$$

then Theorem 2.1 implies the convergence of the corresponding quasi–interpolant in the norm of L_p with the rate $(h\sqrt{\mathcal{D}(h)})^N$. Some special results in this direction are obtained in Beatson-Light [3]. But we prefer to consider the case of fixed $\mathcal{D}$ which is advantageous in numerical applications of the approximations (1.4) we are interested in. First of all one can fix $\mathcal{D}$ such that any prescribed accuracy can be reached with the approximation rate h^N. Further, the cubature of convolution operators $\mathcal{A}$ requires similar to (3.2) the evaluation of $\mathcal{A}\,\eta((\mathbf{x} - h\mathbf{m})/\sqrt{\mathcal{D}}h)$. On the grid points $h\mathbf{k}$, $\mathbf{k} \in \mathbf{Z}^n$, one has to determine the values $\mathcal{A}\,\eta((\mathbf{k} - \mathbf{m})/\sqrt{\mathcal{D}})$, which can be used for any gridsize h. Therefore it is very efficient to precompute and store these values if the cubature is a part of some iterative or multiscale algorithm.

In order to derive an analytic expression for $\mathcal{L}_n \eta_{2M}(\mathbf{x})$ we use the representation

$$L_{M-1}^{(n/2)}(|\mathbf{x}|^2)\, e^{-|\mathbf{x}|^2} = \sum_{j=0}^{M-1} \frac{(-1)^j}{j!\, 4^j}\, \Delta^j e^{-|\mathbf{x}|^2} \;, \quad \mathbf{x} \in \mathbf{R^n} \;, \tag{3.7}$$

(see [7]), which together with the summation formula

$$L_j^{(\alpha-1)}(y) = L_j^{(\alpha)}(y) - L_{j-1}^{(\alpha)}(y) \tag{3.8}$$

(Abramowitz-Stegun [1], 22.7.30) leads to the equality

$$L_j^{(n/2-1)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2} = \frac{(-1)^j}{j!\, 4^j}\, \Delta^j e^{-|\mathbf{x}|^2} \;, \quad \mathbf{x} \in \mathbf{R^n} \;. \tag{3.9}$$

Now we are in the position to determine the integral

$$\mathcal{L}_n \big(L_{M-1}^{(n/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathbf{x}) = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \int\limits_{\mathbf{R^n}} \frac{L_{M-1}^{(n/2)}(|\mathbf{y}|^2)\, e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|^{n-2}}\, d\mathbf{y} \;, \tag{3.10}$$

Since $\mathcal{L}_n \Delta = -I$ the representations (3.7) and (3.9) lead to

$$\begin{aligned} \mathcal{L}_n \big(L_{M-1}^{(n/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathbf{x}) &= \mathcal{L}_n \big(e^{-|\cdot|^2}\big)(\mathbf{x}) - \sum_{j=1}^{M-1} \frac{(-1)^j}{j!\, 4^j} \Delta^{j-1} e^{-|\mathbf{x}|^2} \\ &= \mathcal{L}_n \big(e^{-|\cdot|^2}\big)(\mathbf{x}) + \sum_{j=1}^{M-1} \frac{1}{4\, j} \frac{(-1)^{j-1}}{(j-1)!\, 4^{j-1}} \Delta^{j-1} e^{-|\mathbf{x}|^2} \\ &= \mathcal{L}_n \big(e^{-|\cdot|^2}\big)(\mathbf{x}) + e^{-|\mathbf{x}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(n/2-1)}(|\mathbf{x}|^2)}{4(j+1)} \;. \end{aligned} \tag{3.11}$$

Thus it remains to determine $\mathcal{L}_n \big(e^{-|\cdot|^2}\big)(\mathbf{x})$. We use that

$$\mathcal{L}_n \big(e^{-|\cdot|^2}\big)(\mathbf{x}) = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \int\limits_{\mathbf{R^n}} \frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|^{n-2}}\, d\mathbf{y} = \pi^{n/2} \int\limits_{\mathbf{R^n}} \frac{e^{-\pi^2|\lambda|^2}}{4\pi^2|\lambda|^2}\, e_{\boldsymbol{\lambda}}(\mathbf{x})\, d\lambda$$

$$= \frac{2\pi^{n/2+1}}{|\mathbf{x}|^{n/2-1}} \int\limits_0^\infty \frac{e^{-\pi^2 r^2}}{4\pi^2 r^2} J_{n/2-1}(2\pi r|\mathbf{x}|) r^{n/2} dr = \frac{1}{4|\mathbf{x}|^{n-2}} \int\limits_0^{|\mathbf{x}|^2} \tau^{n/2-2} e^{-\tau}\, d\tau \;,$$

(see Stein-Weiss [10], Th.IV.3.3, Bateman-Erdélyi [2], 8.6.11).

In particular,

$$\mathcal{L}_4 \big(e^{-|\cdot|^2}\big)(\mathbf{x}) = \frac{1}{4\pi^2} \int\limits_{\mathbf{R^4}} \frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|^2}\, d\mathbf{y} = \frac{1}{4|\mathbf{x}|^2} \big(1 - e^{-|\mathbf{x}|^2}\big)$$

and

$$\mathcal{L}_3 \big(e^{-|\cdot|^2}\big)(\mathbf{x}) = \frac{1}{4\pi} \int\limits_{\mathbf{R^3}} \frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y} = \frac{1}{2|\mathbf{x}|} \int\limits_0^{|\mathbf{x}|} e^{-\tau^2}\, d\tau = \frac{\sqrt{\pi}}{4|\mathbf{x}|}\, \mathrm{erf}\,(|\mathbf{x}|) \tag{3.12}$$

with the error function erf.

Noting the relation to Hermite polynomials

$$L_j^{(1/2)}(y) = \frac{(-1)^j}{j!\, 2^{2j+1}\sqrt{y}}\, H_{2j+1}(\sqrt{y}) \;, \quad H_j(y) := (-1)^j\, e^{y^2} \Big(\frac{d}{dy}\Big)^j e^{-y^2} \;,$$

from (3.11) we obtain in the special case $n = 3$

$$\mathcal{L}_3 \eta_{2M}(\mathrm{x}) = \frac{1}{4\pi^{3/2}|\mathrm{x}|}\Big(\sqrt{\pi}\,\mathrm{erf}\,(|\mathrm{x}|) + e^{-|\mathrm{x}|^2} \sum_{j=0}^{M-2} \frac{(-1)^j\, H_{2j+1}(|\mathrm{x}|)}{(j+1)!\, 2^{2j+1}}\Big)\,. \tag{3.13}$$

The described and related cubature formulas were tested in a large number of examples and partly included in numerical algorithms for solving partial integro-differential equations (see [6]). Let us provide the results of two tests with the three–dimensional cubature formula (3.6), (3.13). In Fig. 1 we have plotted the relative cubature error for different generating functions η_{2M} and sufficiently smooth densities. The corresponding values of the approximation rate

$$(\log|\mathcal{L}_3 u(0) - \mathcal{L}_{3,2h}u(0)| - \log|\mathcal{L}_3 u(0) - \mathcal{L}_{3,h}u(0)|)/\log 2$$

are contained in Table 1. It shows that pointwise the approximation rate can even be higher than theoretically predicted. For example, in all numerical tests we obtained an approximation rate near 6 for the density $(1-|\mathrm{x}|^2)_+^4$ having discontinuous fourth derivatives.

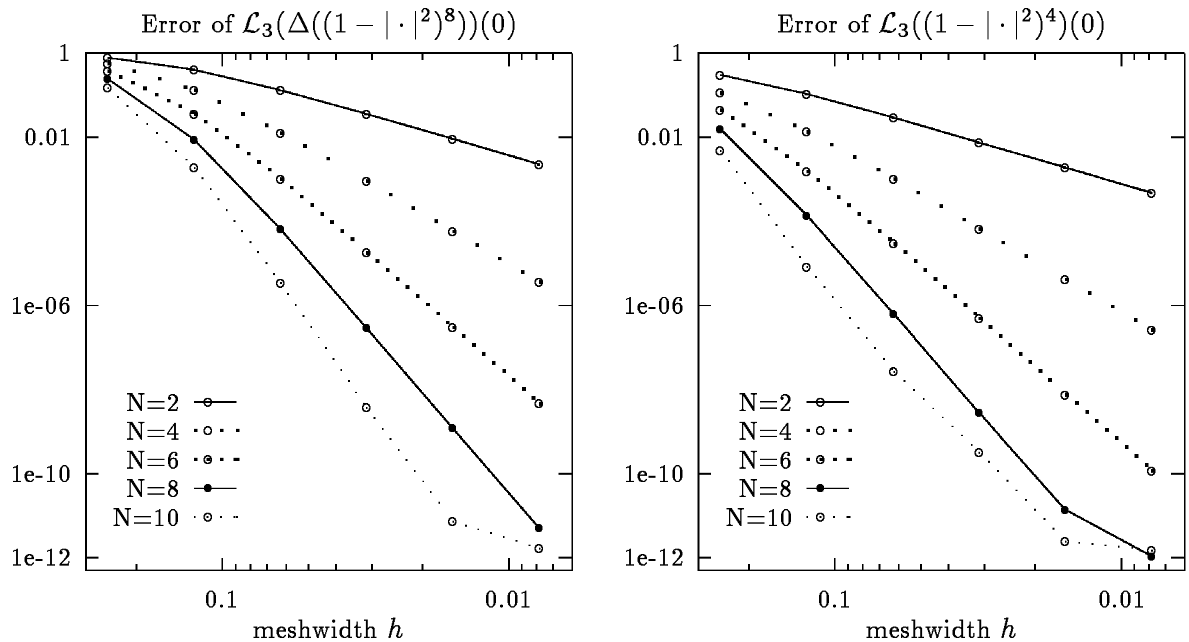

FIGURE 1. Cubature error for the Newton potential using different generating functions η_{2M} and the fixed parameter $\mathcal{D} = 3$.

h^{-1}		$\mathcal{L}_3(\Delta((1-\lvert\cdot\rvert^2)_+^8))$					$\mathcal{L}_3((1-\lvert\cdot\rvert^2)_+^4)$				
	M	1	2	3	4	5	1	2	3	4	5
8		0.975	2.117	3.383	4.780	6.338	1.525	3.088	4.817	6.830	9.227
16		1.619	3.354	5.176	7.087	9.098	1.857	3.750	5.722	7.764	8.282
32		1.893	3.826	5.787	7.775	9.810	1.963	3.937	5.933	7.819	6.344
64		1.973	3.956	5.947	7.946	9.033	1.991	3.984	5.983	7.639	7.024
128		1.993	3.989	5.986	7.851	2.124	1.998	3.996	6.009	3.727	0.715

TABLE 1. Approximation rate of the cubature at the point $\mathrm{x} = 0$ for the densities $\Delta((1-|\mathrm{x}|^2)_+^8)$ and $(1-|\mathrm{x}|^2)_+^4$, resp., using different functions η_{2M} and $\mathcal{D} = 3$.

In the case $n=2$ we have

$$\mathcal{L}_2 u(\mathbf{x}) = \frac{1}{2\pi} \int\limits_{\mathbf{R}^2} \log \frac{1}{|\mathbf{x}-\mathbf{y}|}\, u(\mathbf{y})\, d\mathbf{y} \;, \tag{3.14}$$

and one easily obtains

$$\mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x}) = -\frac{1}{4}\,\mathrm{E}_1(|\mathbf{x}|^2) - \frac{1}{2}\log|\mathbf{x}| = \frac{1}{4}\Big(\gamma - \int\limits_0^{|\mathbf{x}|^2} \frac{1-e^{-\tau}}{\tau}\, d\tau\Big) \;, \tag{3.15}$$

where E_1 is the exponential integral and γ is the Euler constant $\gamma = 0.577215\ldots$. Thus from formula (3.11) we derive the analytic expression

$$\mathcal{L}_2\eta_{2M}(\mathbf{x}) = \frac{1}{4\pi}\Big(2\log\frac{1}{|\mathbf{x}|} - \mathrm{E}_1(|\mathbf{x}|^2) + e^{-|\mathbf{x}|^2}\sum_{j=0}^{M-2}\frac{L_j(|\mathbf{x}|^2)}{(j+1)}\Big) \tag{3.16}$$

with the Laguerre polynomials $L_j = L_j^{(0)}$, leading to the cubature formula

$$\mathcal{L}_{2,h}u(\mathbf{x}) = h^2 \sum_{\mathbf{m}\in\mathbf{Z}^2} u(h\mathbf{m})\Big(\mathcal{L}_2\eta_{2M}\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) - \frac{\log\sqrt{\mathcal{D}}h}{2\pi}\Big) \;.$$

4. Elastic and Diffraction Potentials

In this section we derive analytic formulas of elastic and diffraction potentials applied to the generating functions η_{2M} defined in (3.3). It is clear that analogously to the preceding section these formulas can be used to construct high order cubature formulas for the corresponding potentials. Based on the mapping properties of the integral operators estimates of the cubature error can be obtained similar to those of Theorems 3.1 and 3.2.

4.1. **2d-elastic potential.** A solution of the two–dimensional Lamé system

$$\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\nabla\,\mathrm{div}\,\mathbf{u} = \mathbf{f}$$

is given by

$$u_k(\mathbf{x}) = \int\limits_{\mathbf{R}^2} \Gamma_{kl}(\mathbf{x},\mathbf{y}) f_l(\mathbf{y})\, d\mathbf{y} \;,$$

where for $k,l = 1,2$

$$\Gamma_{kl}(\mathbf{x},\mathbf{y}) = -\frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\Big(\delta_{kl}\frac{\lambda+3\mu}{\lambda+\mu}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_k-y_k)(x_l-y_l)}{|\mathbf{x}-\mathbf{y}|^2}\Big) \;,$$

is the Boussinesq fundamental matrix. In the following we determine

$$\int\limits_{\mathbf{R}^2} \Gamma_{kl}(\mathbf{x},\mathbf{y}) L_{M-1}^{(1)}(|\mathbf{y}|^2)\, e^{-|\mathbf{y}|^2} d\mathbf{y} \;.$$

Note that

$$\Gamma_{kl}(\mathbf{x},\mathbf{y}) = -\frac{\delta_{kl}}{2\pi\mu}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{\partial^2}{\partial x_k \partial x_l}\Big(|\mathbf{x}-\mathbf{y}|^2\Big(\log\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{1}{2}\Big)\Big) \;.$$

From (3.14) and (3.16) it is clear that it remains to determine the integrals

$$\begin{aligned}
I_{kl} :=& \frac{\partial^2}{\partial x_k \partial x_l} \int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, L^{(1)}_{M-1}(|\mathrm{y}|^2)\, e^{-|\mathrm{y}|^2} d\mathbf{y} \\
=& \frac{\partial^2}{\partial x_k \partial x_l} \int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, e^{-|\mathrm{y}|^2} d\mathbf{y} \\
&+ \sum_{j=1}^{M-1} \frac{(-1)^j}{j!\,4^j} \frac{\partial^2}{\partial x_k \partial x_l} \int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, \Delta^j e^{-|\mathrm{y}|^2} d\mathbf{y}\ .
\end{aligned}$$

First we remark that

$$\begin{aligned}
&\frac{\partial}{\partial x_l} \int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, e^{-|\mathrm{y}|^2} d\mathbf{y} \\
&\qquad = \big(2x_l + \frac{\partial}{\partial x_l}\big) \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, e^{-|\mathrm{y}|^2} d\mathbf{y}\ ,
\end{aligned} \tag{4.1}$$

such that for the case $M=1$ we derive

$$\frac{I_{kl}}{2\pi} = 2\,\delta_{kl}\, \mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathrm{x}) + \big(2x_l \frac{\partial}{\partial x_k} + \frac{\partial^2}{\partial x_k \partial x_l}\big) \mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathrm{x})\ ,$$

leading together with (3.15) to the formula

$$\begin{aligned}
\int_{\mathbf{R}^2} \Gamma_{kl}(\mathrm{x},\mathrm{y})\, \eta_2(\mathrm{y})\, d\mathbf{y} =& \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\, \delta_{kl} \big(\ \mathrm{E}_1(|\mathrm{x}|^2) + 2\log|\mathrm{x}|\big) \\
&+ \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)} \Big(\Big(\frac{x_k x_l}{|\mathrm{x}|^2} - \frac{\delta_{kl}}{2}\Big) \frac{1-e^{-|\mathrm{x}|^2}}{|\mathrm{x}|^2} - \frac{x_k x_l}{|\mathrm{x}|^2} \Big)\ .
\end{aligned} \tag{4.2}$$

For the case $M \ge 2$ we use the relation

$$\Delta\, |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big) = 4 \log \frac{1}{|\mathrm{x}-\mathrm{y}|} - 2$$

and Green's second formula to get

$$\begin{aligned}
I_{kl} =& \frac{\partial^2}{\partial x_k \partial x_l} \Big(\int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, e^{-|\mathrm{y}|^2} d\mathbf{y} - \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, e^{-|\mathrm{y}|^2} d\mathbf{y} \Big) \\
&+ \frac{\partial^2}{\partial x_k \partial x_l} \sum_{j=2}^{M-1} \frac{(-1)^j}{j!\,4^j} \int_{\mathbf{R}^2} \big(4 \log \frac{1}{|\mathrm{x}-\mathrm{y}|} - 2\big)\, \Delta^{j-1} e^{-|\mathrm{y}|^2} d\mathbf{y} \\
=& \frac{\partial^2}{\partial x_k \partial x_l} \Big(\int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, e^{-|\mathrm{y}|^2} d\mathbf{y} - \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, e^{-|\mathrm{y}|^2} d\mathbf{y} \Big) \\
&+ \frac{\partial^2}{\partial x_k \partial x_l} \sum_{j=2}^{M-1} \frac{(-1)^j}{j!\,4^{j-1}} \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, \Delta^{j-1} e^{-|\mathrm{y}|^2} d\mathbf{y}\ .
\end{aligned}$$

Now relation (4.1) helps to simplify

$$\begin{aligned}
&\frac{\partial^2}{\partial x_k \partial x_l} \Big(\int_{\mathbf{R}^2} |\mathrm{x}-\mathrm{y}|^2 \big(\log \frac{1}{|\mathrm{x}-\mathrm{y}|} + \frac{1}{2}\big)\, e^{-|\mathrm{y}|^2} d\mathbf{y} - \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, e^{-|\mathrm{y}|^2} d\mathbf{y} \Big) \\
&= 2 \frac{\partial}{\partial x_k} x_l \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, e^{-|\mathrm{y}|^2} d\mathbf{y} = 2\Big(\delta_{kl} + x_l \frac{\partial}{\partial x_k}\Big) \int_{\mathbf{R}^2} \log \frac{1}{|\mathrm{x}-\mathrm{y}|}\, e^{-|\mathrm{y}|^2} d\mathbf{y}\ .
\end{aligned}$$

Since for $j \geq 2$ there holds

$$\int\limits_{\mathbf{R}^2} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \, \Delta^{j-1} e^{-|\mathbf{y}|^2} d\mathbf{y} = -2\pi \, \Delta^{j-2} e^{-|\mathbf{x}|^2} \, ,$$

we derive by using (3.9)

$$I_{kl} = 2\Big(\delta_{kl} + x_l \frac{\partial}{\partial x_k}\Big) \int\limits_{\mathbf{R}^2} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \, e^{-|\mathbf{y}|^2} d\mathbf{y} - \pi \frac{\partial^2}{\partial x_k \partial x_l} \sum_{j=0}^{M-3} \frac{L_j^{(0)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}}{2(j+1)(j+2)} \, ,$$

where of course the sum disappears if $M = 2$. Thus

$$\int\limits_{\mathbf{R}^2} \Gamma_{kl}(\mathbf{x},\mathbf{y}) L_{M-1}^{(1)}(|\mathbf{y}|^2) \, e^{-|\mathbf{y}|^2} d\mathbf{y} = -\frac{\delta_{kl}}{\mu} \mathcal{L}_2\big(L_{M-1}^{(1)}(|\cdot|^2) \, e^{-|\cdot|^2}\big)(\mathbf{x})$$

$$+ \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \bigg(\Big(\delta_{kl} + x_l \frac{\partial}{\partial x_k}\Big) \mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x}) - \sum_{j=0}^{M-3} \frac{\partial^2}{\partial x_k \partial x_l} \frac{L_j^{(0)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}}{8(j+1)(j+2)} \bigg) \, .$$

Now we use that

$$\Big(\frac{d}{dy}\Big)^m \Big(L_j^{(\alpha)}(y) \, e^{-y}\Big) = (-1)^m L_j^{(\alpha+m)}(y) \, e^{-y} \, , \tag{4.3}$$

which follows from the formula

$$(1-z)^{-\alpha-1} e^{-y/(1-z)} = \sum_{j=0}^{\infty} L_j^{(\alpha)}(y) e^{-y} z^j \quad ([1], 22.9.15) \, ,$$

to derive

$$\frac{\partial^2}{\partial x_k \partial x_l} L_j^{(0)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2} = -2\delta_{kl} L_j^{(1)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2} + 4 x_k x_l L_j^{(2)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2} \, ,$$

which leads to the equality

$$\int\limits_{\mathbf{R}^2} \Gamma_{kl}(\mathbf{x},\mathbf{y}) L_{M-1}^{(1)}(|\mathbf{y}|^2) \, e^{-|\mathbf{y}|^2} d\mathbf{y} = \delta_{kl} \bigg(-\frac{1}{\mu} \mathcal{L}_2\big(L_{M-1}^{(1)}(|\cdot|^2) \, e^{-|\cdot|^2}\big)(\mathbf{x})$$

$$+ \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \Big(\mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x}) + \sum_{j=0}^{M-3} \frac{L_j^{(1)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}}{4(j+1)(j+2)} \Big) \bigg)$$

$$+ \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \Big(x_l \frac{\partial}{\partial x_k} \mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x}) - x_k x_l \sum_{j=0}^{M-3} \frac{L_j^{(2)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}}{2(j+1)(j+2)} \Big) \, .$$

Next we simplify the expressions enclosed in the brackets. Using (3.8) one easily transforms

$$\sum_{j=0}^{M-3} \frac{L_j^{(1)}(|\mathbf{x}|^2)}{4(j+1)(j+2)} = \sum_{j=0}^{M-2} \frac{L_j^{(0)}(|\mathbf{x}|^2)}{4(j+1)} - \frac{L_{M-2}^{(1)}(|\mathbf{x}|^2)}{4(M-1)} \, , \tag{4.4}$$

such that from (3.11) we get

$$\mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x}) + \sum_{j=0}^{M-3} \frac{L_j^{(1)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}}{4(j+1)(j+2)}$$

$$= \mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x}) + e^{-|\mathbf{x}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(0)}(|\mathbf{x}|^2)}{4(j+1)} - \frac{L_{M-2}^{(1)}(|\mathbf{x}|^2)}{4(M-1)} \, e^{-|\mathbf{x}|^2}$$

$$= \mathcal{L}_2\big(L_{M-1}^{(1)}(|\cdot|^2) \, e^{-|\cdot|^2}\big)(\mathbf{x}) - \frac{L_{M-2}^{(1)}(|\mathbf{x}|^2)}{4(M-1)} \, e^{-|\mathbf{x}|^2} \, ,$$

implying for the first term

$$
-\frac{1}{\mu}\mathcal{L}_2\big(L_{M-1}^{(1)}(|\cdot|^2)\,e^{-|\cdot|^2}\big)(\mathbf{x})+\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\Big(\mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x})+\sum_{j=0}^{M-3}\frac{L_j^{(1)}(|\mathbf{x}|^2)e^{-|\mathbf{x}|^2}}{4(j+1)(j+2)}\Big)
$$

$$
=-\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\mathcal{L}_2\big(L_{M-1}^{(1)}(|\cdot|^2)\,e^{-|\cdot|^2}\big)(\mathbf{x})-\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{L_{M-2}^{(1)}(|\mathbf{x}|^2)}{4(M-1)}\,e^{-|\mathbf{x}|^2}\ .
$$

Now we consider the second expression. From the relation

$$
L_j^{(\alpha+1)}(y)=\frac{1}{y}\Big((j+\alpha+1)\,L_j^{(\alpha)}(y)-(j+1)\,L_{j+1}^{(\alpha)}(y)\Big) \tag{4.5}
$$

(see Abramowitz-Stegun [1], 22.7.31) we get

$$
L_j^{(2)}(|\mathbf{x}|^2)=\frac{1}{|\mathbf{x}|^2}\big((j+2)L_j^{(1)}(|\mathbf{x}|^2)-(j+1)L_{j+1}^{(1)}(|\mathbf{x}|^2)\big)\ ,
$$

and therefore

$$
x_kx_l\sum_{j=0}^{M-3}\frac{L_j^{(2)}(|\mathbf{x}|^2)}{2(j+1)(j+2)}=\frac{x_kx_l}{|\mathbf{x}|^2}\sum_{j=0}^{M-3}\frac{(j+2)L_j^{(1)}(|\mathbf{x}|^2)-(j+1)L_{j+1}^{(1)}(|\mathbf{x}|^2)}{2(j+1)(j+2)}
$$

$$
=\frac{x_kx_l}{|\mathbf{x}|^2}\sum_{j=0}^{M-3}\Big(\frac{L_j^{(1)}(|\mathbf{x}|^2)}{2(j+1)}-\frac{L_{j+1}^{(1)}(|\mathbf{x}|^2)}{2(j+2)}\Big)=\frac{x_kx_l}{2|\mathbf{x}|^2}\Big(1-\frac{L_{M-2}^{(1)}(|\mathbf{x}|^2)}{M-1}\Big)\ .
$$

Furthermore, (3.15) shows that

$$
x_l\frac{\partial}{\partial x_k}\mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x})=\frac{x_l}{4}\frac{\partial}{\partial x_k}\Big(\gamma-\int_0^{|\mathbf{x}|^2}\frac{1-e^{-\tau}}{t}d\tau\Big)=\frac{x_kx_l}{2|\mathbf{x}|^2}(e^{-|\mathbf{x}|^2}-1)\ ,
$$

hence we derive

$$
x_l\frac{\partial}{\partial x_k}\mathcal{L}_2\big(e^{-|\cdot|^2}\big)(\mathbf{x})-x_kx_l\sum_{j=0}^{M-3}\frac{L_j^{(2)}(|\mathbf{x}|^2)e^{-|\mathbf{x}|^2}}{2(j+1)(j+2)}=\frac{x_kx_l}{2|\mathbf{x}|^2}\Big(\frac{L_{M-2}^{(1)}(|\mathbf{x}|^2)}{M-1}\,e^{-|\mathbf{x}|^2}-1\Big)\ .
$$

Consequently, the integration of the generating function η_{2M} results in

$$
\begin{aligned}
\int_{\mathbf{R}^2}\Gamma_{kl}(\mathbf{x},\mathbf{y})\,\eta_{2M}(\mathbf{y})\,d\mathbf{y}&=-\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\,\delta_{kl}\,\mathcal{L}_2\eta_{2M}(\mathbf{x})\\
&+\frac{\lambda+\mu}{4\mu(\lambda+2\mu)}\Big(\Big(\frac{x_kx_l}{|\mathbf{x}|^2}-\frac{\delta_{kl}}{2}\Big)\frac{\eta_{2M-2}(\mathbf{x})}{M-1}-\frac{x_kx_l}{\pi\,|\mathbf{x}|^2}\Big)\ .
\end{aligned} \tag{4.6}
$$

We note that by simple differentiation of the expressions (4.2) and (4.6) it is possible to obtain effective approximations of the stress tensor

$$
\sigma_k=\lambda\big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\big)+2\mu\frac{\partial u_k}{\partial x_k}\quad,\quad\tau_{1,2}=\mu\big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\big)\ .
$$

4.2. **3d-elastic potential.** A solution of the Lamé system

$$
\mu\,\Delta\mathbf{u}+(\lambda+\mu)\,\nabla\ \mathrm{div}\ \mathbf{u}=\mathbf{f}
$$

can be obtained from

$$
u_k(\mathbf{x})=\int_{\mathbf{R}^3}\Gamma_{kl}(\mathbf{x},\mathbf{y})f_l(\mathbf{y})\,d\mathbf{y}\ ,
$$

where the elements of the Kelvin-Somigliana fundamental matrix $\{\Gamma_{kl}\}_{k,l=1}^{3}$ are given by

$$\Gamma_{kl}(\mathbf{x},\mathbf{y}) = -\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\left(\frac{\lambda+3\mu}{\lambda+\mu}\frac{\delta_{kl}}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_k-y_k)(x_l-y_l)}{|\mathbf{x}-\mathbf{y}|^3}\right),$$

The integrals

$$\int_{\mathbf{R}^3} \Gamma_{kl}(\mathbf{x},\mathbf{y})\,\eta_{2M}(\mathbf{y})\,d\mathbf{y} = \pi^{-3/2}\int_{\mathbf{R}^3} \Gamma_{kl}(\mathbf{x},\mathbf{y}) L_{M-1}^{(3/2)}(|\mathbf{y}|^2)\,e^{-|\mathbf{y}|^2}\,d\mathbf{y}$$

can be determined quite similar to the preceding subsection. First we note that

$$\Gamma_{kl}(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi\mu}\frac{\delta_{kl}}{|\mathbf{x}-\mathbf{y}|} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{\partial^2}{\partial x_k \partial x_l}|\mathbf{x}-\mathbf{y}|\,.$$

Hence it remains to determine the integrals

$$\begin{aligned}
I_{kl} :=& \frac{\partial^2}{\partial x_k \partial x_l}\int_{\mathbf{R}^3} |\mathbf{x}-\mathbf{y}| L_{M-1}^{(3/2)}(|\mathbf{y}|^2)\,e^{-|\mathbf{y}|^2}\,d\mathbf{y} \\
=& \frac{\partial^2}{\partial x_k \partial x_l}\int_{\mathbf{R}^3} |\mathbf{x}-\mathbf{y}| e^{-|\mathbf{y}|^2}\,d\mathbf{y} + \sum_{j=1}^{M-1}\frac{(-1)^j}{j!\,4^j}\frac{\partial^2}{\partial x_k \partial x_l}\int_{\mathbf{R}^3} |\mathbf{x}-\mathbf{y}|\Delta^j e^{-|\mathbf{y}|^2}\,d\mathbf{y}\,.
\end{aligned}$$

Since for $j \geq 2$ there holds

$$\int_{\mathbf{R}^3} |\mathbf{x}-\mathbf{y}|\,\Delta^j e^{-|\mathbf{y}|^2}\,d\mathbf{y} = 2\,\Delta^{j-1}\int_{\mathbf{R}^3}\frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} = -8\pi\,\Delta^{j-2}e^{-|\mathbf{x}|^2}\,,$$

the desired integral equals

$$\begin{aligned}
I_{kl} =& \frac{\partial^2}{\partial x_k \partial x_l}\left(\int_{\mathbf{R}^3} |\mathbf{x}-\mathbf{y}|\,e^{-|\mathbf{y}|^2}\,d\mathbf{y} - \frac{1}{2}\int_{\mathbf{R}^3}\frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\right) \\
&- 8\pi\frac{\partial^2}{\partial x_k \partial x_l}\sum_{j=2}^{M-1}\frac{(-1)^j}{j!\,4^j}\Delta^{j-2}e^{-|\mathbf{x}|^2}\,.
\end{aligned}$$

Similar to (4.1) we have

$$\frac{\partial}{\partial x_l}\int_{\mathbf{R}^3} |\mathbf{x}-\mathbf{y}|\,e^{-|\mathbf{y}|^2}\,d\mathbf{y} = (x_l + \frac{1}{2}\frac{\partial}{\partial x_l})\int_{\mathbf{R}^3}\frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,, \tag{4.7}$$

implying for $M=1$ the expression

$$\begin{aligned}
\int_{\mathbf{R}^3} \Gamma_{kl}(\mathbf{x},\mathbf{y})\,\eta_2(\mathbf{y})\,d\mathbf{y} =& -\frac{\lambda+3\mu}{2\pi\mu(\lambda+2\mu)}\frac{\operatorname{erf}(|\mathbf{x}|)}{4|\mathbf{x}|}\,\delta_{kl} \\
&+ \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\big(2x_l\frac{\partial}{\partial x_k} + \frac{\partial^2}{\partial x_k \partial x_l}\big)\frac{\operatorname{erf}(|\mathbf{x}|)}{4|\mathbf{x}|} \\
=& \frac{(\lambda+\mu)\,x_k x_l}{16\pi\mu(\lambda+2\mu)}\Big(\frac{(3-2|\mathbf{x}|^2)\operatorname{erf}(|\mathbf{x}|)}{|\mathbf{x}|^5} - \frac{6\,e^{-|\mathbf{x}|^2}}{\sqrt{\pi}\,|\mathbf{x}|^4}\Big) \\
&+ \frac{\delta_{kl}}{16\pi\mu(\lambda+2\mu)}\Big((\lambda+\mu)\Big(\frac{2\,e^{-|\mathbf{x}|^2}}{\sqrt{\pi}\,|\mathbf{x}|^2} - \frac{\operatorname{erf}(|\mathbf{x}|)}{|\mathbf{x}|^3}\Big) - 2(\lambda+3\mu)\frac{\operatorname{erf}(|\mathbf{x}|)}{|\mathbf{x}|}\Big)\,.
\end{aligned} \tag{4.8}$$

If $M \geq 2$ then (4.7) yields

$$I_{kl} = \Big(\delta_{kl} + x_l\frac{\partial}{\partial x_k}\Big)\int_{\mathbf{R}^3}\frac{e^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} - 8\pi\frac{\partial^2}{\partial x_k \partial x_l}\sum_{j=0}^{M-3}\frac{L_j^{(1/2)}(|\mathbf{x}|^2)e^{-|\mathbf{x}|^2}}{16(j+1)(j+2)}\,,$$

such that

$$\int\limits_{\mathbf{R}^3} \Gamma_{kl}(\mathrm{x},\mathrm{y}) L_{M-1}^{(3/2)}(|\mathrm{y}|^2)\, e^{-|\mathrm{y}|^2}\, d\mathrm{y} = -\frac{\delta_{kl}}{\mu} \mathcal{L}_3\big(L_{M-1}^{(3/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathrm{x})$$

$$+ \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \bigg(\Big(\delta_{kl} + x_l \frac{\partial}{\partial x_k}\Big) \mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) - \sum_{j=0}^{M-3} \frac{\partial^2}{\partial x_k \partial x_l} \frac{L_j^{(1/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{8(j+1)(j+2)} \bigg) \,.$$

From (4.3) we see that

$$\frac{\partial^2}{\partial x_k \partial x_l} L_j^{(1/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2} = -2\delta_{kl} L_j^{(3/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2} + 4 x_k x_l L_j^{(5/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}$$

which leads to

$$\int\limits_{\mathbf{R}^3} \Gamma_{kl}(\mathrm{x},\mathrm{y}) L_{M-1}^{(3/2)}(|\mathrm{y}|^2)\, e^{-|\mathrm{y}|^2}\, d\mathrm{y} = \delta_{kl} \bigg(-\frac{1}{\mu} \mathcal{L}_3\big(L_{M-1}^{(3/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathrm{x})$$

$$+ \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \Big(\mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) + \sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{4(j+1)(j+2)} \Big) \bigg)$$

$$+ \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \Big(x_l \frac{\partial}{\partial x_k} \mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) - x_k x_l \sum_{j=0}^{M-3} \frac{L_j^{(5/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{2(j+1)(j+2)} \Big) \,.$$

Again we consider the separate terms. Similar to (4.4) we derive

$$\sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2)}{4(j+1)(j+2)} = \sum_{j=0}^{M-2} \frac{L_j^{(1/2)}(|\mathrm{x}|^2)}{4(j+1)} - \frac{L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{4(M-1)} \,, \tag{4.9}$$

such that from (3.11) we have

$$\mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) + \sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{4(j+1)(j+2)}$$

$$= \mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) + e^{-|\mathrm{x}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(1/2)}(|\mathrm{x}|^2)}{4(j+1)} - \frac{L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{4(M-1)}\, e^{-|\mathrm{x}|^2}$$

$$= \mathcal{L}_3\big(L_{M-1}^{(3/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathrm{x}) - \frac{L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{4(M-1)}\, e^{-|\mathrm{x}|^2} \,,$$

which implies

$$-\frac{1}{\mu} \mathcal{L}_3\big(L_{M-1}^{(3/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathrm{x}) + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \Big(\mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) + \sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{4(j+1)(j+2)} \Big)$$

$$= -\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \mathcal{L}_3\big(L_{M-1}^{(3/2)}(|\cdot|^2)\, e^{-|\cdot|^2}\big)(\mathrm{x}) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{L_{M-2}^{(3/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{4(M-1)} \,.$$

To handle the second term we use (4.5) to write

$$L_j^{(5/2)}(|\mathrm{x}|^2) = \frac{1}{|\mathrm{x}|^2} \Big((j + \frac{5}{2}) L_j^{(3/2)}(|\mathrm{x}|^2) - (j+1) L_{j+1}^{(3/2)}(|\mathrm{x}|^2) \Big) \,,$$

hence we get

$$x_k x_l \sum_{j=0}^{M-3} \frac{L_j^{(5/2)}(|\mathrm{x}|^2)}{2(j+1)(j+2)}$$

$$= \frac{x_k x_l}{|\mathrm{x}|^2}\Big(\sum_{j=0}^{M-3} \Big(\frac{L_j^{(3/2)}(|\mathrm{x}|^2)}{2(j+1)} - \frac{L_{j+1}^{(3/2)}(|\mathrm{x}|^2)}{2(j+2)} \Big) + \sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2)}{4(j+1)(j+2)} \Big)$$

$$= \frac{x_k x_l}{|\mathrm{x}|^2}\Big(\frac{1}{2} - \frac{L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{2(M-1)} + \sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2)}{4(j+1)(j+2)} \Big) .$$

From (3.12) we have

$$x_l \frac{\partial}{\partial x_k} \mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) = x_l \frac{\partial}{\partial x_k} \frac{1}{2|\mathrm{x}|} \int_0^{|\mathrm{x}|} e^{-\tau^2}\, d\tau = -\frac{x_k x_l}{2|\mathrm{x}|^3} \int_0^{|\mathrm{x}|} e^{-\tau^2}\, d\tau + \frac{x_k x_l}{2|\mathrm{x}|^2} e^{-|\mathrm{x}|^2} ,$$

therefore the second term transforms to

$$x_l \frac{\partial}{\partial x_k} \mathcal{L}_3\big(e^{-|\cdot|^2}\big)(\mathrm{x}) - x_k x_l \sum_{j=0}^{M-3} \frac{L_j^{(5/2)}(|\mathrm{x}|^2) e^{-|\mathrm{x}|^2}}{2(j+1)(j+2)}$$

$$= -\frac{x_k x_l}{2|\mathrm{x}|^3} \int_0^{|\mathrm{x}|} e^{-\tau^2}\, d\tau + \frac{x_k x_l}{2|\mathrm{x}|^2} e^{-|\mathrm{x}|^2}$$

$$- \frac{x_k x_l}{|\mathrm{x}|^2}\Big(\frac{1}{2} - \frac{L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{2(M-1)} + \sum_{j=0}^{M-3} \frac{L_j^{(3/2)}(|\mathrm{x}|^2)}{4(j+1)(j+2)} \Big) e^{-|\mathrm{x}|^2}$$

$$= -\frac{x_k x_l}{|\mathrm{x}|^2}\Big(\frac{1}{2|\mathrm{x}|} \int_0^{|\mathrm{x}|} e^{-\tau^2}\, d\tau + e^{-|\mathrm{x}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(1/2)}(|\mathrm{x}|^2)}{4(j+1)} - \frac{3\, L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{4(M-1)} e^{-|\mathrm{x}|^2} \Big)$$

$$= \frac{x_k x_l}{|\mathrm{x}|^2} \Big(\frac{3\, L_{M-2}^{(3/2)}(|\mathrm{x}|^2)}{4(M-1)} e^{-|\mathrm{x}|^2} - \mathcal{L}_3\big(L_{M-1}^{(3/2)}(|\cdot|^2) e^{-|\cdot|^2} \big)(\mathrm{x}) \Big) ,$$

where we have used (3.11), (3.12) and (4.9). So we come finally to the expression

$$\int_{\mathbf{R}^3} \Gamma_{kl}(\mathrm{x},\mathrm{y}) \eta_{2M}(\mathrm{y})\, d\mathrm{y} = \frac{\lambda+\mu}{8\mu(\lambda+2\mu)} \Big(3 \frac{x_k x_l}{|\mathrm{x}|^2} - \delta_{kl} \Big) \frac{\eta_{2M-2}(\mathrm{x})}{M-1}$$
$$- \Big(\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\, \delta_{kl} + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{x_k x_l}{|\mathrm{x}|^2} \Big) \mathcal{L}_3 \eta_{2M}(\mathrm{x}) , \tag{4.10}$$

which is valid for $M \geq 2$ and even simpler than the corresponding formula (4.8) for the cubature of second order.

4.3. **Diffraction potential.** Consider the potential

$$\mathcal{S}u(\mathrm{x}) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{ik|\mathrm{x}-\mathrm{y}|}}{|\mathrm{x}-\mathrm{y}|} u(\mathrm{y})\, d\mathrm{y} , \quad \mathrm{Im}\ k \geq 0 \ , \ \mathrm{x} \in \mathbf{R}^3 , \tag{4.11}$$

providing the solution of the Helmholtz equation

$$-(\Delta v + k^2 v) = u ,$$

which satisfies Sommerfeld's radiation condition

$$\big\langle \frac{\mathrm{x}}{|\mathrm{x}|}, \nabla v(\mathrm{x}) \big\rangle - ik\, v(\mathrm{x}) = o\big(\frac{1}{|\mathrm{x}|}\big) , \quad |\mathrm{x}| \to \infty .$$

From the equality

$$(\Delta+k^2)^{-1}\Delta^j=(-1)^j k^{2j}(\Delta+k^2)^{-1}+\sum_{r=0}^{j-1}(-1)^r k^{2r}\Delta^{j-1-r}$$

we derive

$$\begin{aligned}
&\mathcal{S}\big(L^{(3/2)}_{M-1}(|\cdot|^2)\,e^{-|\cdot|^2}\big)(\mathbf{x})=-(\Delta+k^2)^{-1}\Big(\sum_{j=0}^{M-1}\frac{(-1)^j}{j!\,4^j}\Delta^j e^{-|\mathbf{x}|^2}\Big)\\
&=\mathcal{S}\big(e^{-|\cdot|^2}\big)(\mathbf{x})\sum_{j=0}^{M-1}\frac{1}{j!}\Big(\frac{k^2}{4}\Big)^j-\sum_{j=1}^{M-1}\frac{(-1)^j}{j!\,4^j}\sum_{r=0}^{j-1}(-1)^r k^{2r}\Delta^{j-1-r}e^{-|\mathbf{x}|^2}\\
&=\mathcal{S}\big(e^{-|\cdot|^2}\big)(\mathbf{x})\sum_{j=0}^{M-1}\frac{k^{2j}}{j!\,2^{2j}}+e^{-|\mathbf{x}|^2}\sum_{j=0}^{M-2}\frac{1}{4(j+1)!}\sum_{r=0}^{j}r!\Big(\frac{k^2}{4}\Big)^{j-r}L^{(1/2)}_r(|\mathbf{x}|^2)\\
&=\mathcal{S}\big(e^{-|\cdot|^2}\big)(\mathbf{x})\sum_{j=0}^{M-1}\frac{k^{2j}}{j!\,2^{2j}}\\
&\qquad+e^{-|\mathbf{x}|^2}\sum_{j=0}^{M-2}(-1)^j\frac{H_{2j+1}(|\mathbf{x}|)}{2|\mathbf{x}|}\sum_{r=0}^{M-2-j}\frac{k^{2r}}{(j+r+1)!\,2^{2(j+r+1)}}\,.
\end{aligned}$$

Let $\mathrm{Im}\, k>0$. Using the Fourier transform

$$\frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\,e_{\boldsymbol{\lambda}}(-\mathbf{x})\,d\mathbf{x}=\frac{1}{4\pi^2|\lambda|^2-k^2}$$

the convolution equals to

$$\begin{aligned}
\frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,e^{-|\mathbf{y}|^2}\,d\mathbf{y}&=\pi^{3/2}\int_{\mathbf{R}^3}\frac{e^{-\pi^2|\lambda|^2}}{4\pi^2|\lambda|^2-k^2}\,e_{\boldsymbol{\lambda}}(\mathbf{x})\,d\lambda\\
&=\frac{2\pi^{3/2}}{|\mathbf{x}|}\int_0^\infty\frac{\rho e^{-\pi^2\rho^2}}{4\pi^2\rho^2-k^2}\,\sin\,(2\pi\rho|\mathbf{x}|)\,d\rho \qquad (4.12)\\
&=\frac{\sqrt{\pi}}{8|\mathbf{x}|}e^{-k^2/4}\Big(e^{ik|\mathbf{x}|}\mathrm{erfc}\,\big(-|\mathbf{x}|-\frac{ik}{2}\big)-e^{-ik|\mathbf{x}|}\mathrm{erfc}\,\big(|\mathbf{x}|-\frac{ik}{2}\big)\Big)\ ,
\end{aligned}$$

(Bateman-Erdélyi [2], 2.4.26), where the complementary error function is defined as $\mathrm{erfc}\,(z)=1-\mathrm{erf}\,(z)$. Thus we obtain

$$\frac{1}{4\pi^{5/2}}\int_{\mathbf{R}^3}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,e^{-|\mathbf{y}|^2}\,d\mathbf{y}=\frac{e^{-k^2/4}}{4\pi|\mathbf{x}|}\Big(i\sin(k|\mathbf{x}|)+\mathrm{Re}\Big(e^{ik|\mathbf{x}|}\mathrm{erf}\,\big(|\mathbf{x}|+\frac{ik}{2}\big)\Big)\Big)\,.$$

For $k>0$ the Fourier transform of the fundamental solution is the distribution

$$\frac{1}{4\pi^2|\lambda|^2-k^2}+\frac{i\pi}{2k}\,\delta\big(|\lambda|-\frac{k}{2\pi}\big)$$

Hence the convolution (4.12) is the sum of the principal value integral

$$\int_{\mathbf{R}^3}\frac{e^{-\pi^2|\lambda|^2}}{4\pi^2|\lambda|^2-k^2}e^{2\pi i\langle\mathbf{x},\lambda\rangle}d\lambda=\frac{e^{-k^2/4}}{4\pi|\mathbf{x}|}\,\mathrm{Re}\Big(e^{ik|\mathbf{x}|}\mathrm{erf}\,\big(|\mathbf{x}|+\frac{ik}{2}\big)\Big)\ ,$$

where one has to apply the Sochotzkij-Plemelj formula, and of

$$\frac{i\pi}{2k}\Big(\delta\big(|\cdot|-\frac{k}{2\pi}\big),e^{-\pi^2|\cdot|^2}e^{2\pi i\langle\mathbf{x},\cdot\rangle}\Big)=\frac{i}{4\pi}\,\frac{\sin(k|\mathbf{x}|)}{|\mathbf{x}|}\,e^{-k^2/4}\,.$$

Hence we derive also for $k > 0$

$$\pi^{-3/2}\mathcal{S}\left(e^{-|\cdot|^2}\right)(\mathrm{x}) = \frac{1}{4\pi|\mathrm{x}|}e^{-k^2/4}\left(i\sin(k|\mathrm{x}|) + \mathrm{Re}\left(e^{ik|\mathrm{x}|}\mathrm{erf}\left(|\mathrm{x}| + \frac{ik}{2}\right)\right)\right) .$$

Summing up we get the analytic formula

$$\mathcal{S}\eta_{2M}(\mathrm{x}) = \frac{e^{-k^2/4}}{4\pi|\mathrm{x}|}\left(i\sin(k|\mathrm{x}|) + \mathrm{Re}\left(e^{ik|\mathrm{x}|}\mathrm{erf}\left(|\mathrm{x}| + \frac{ik}{2}\right)\right)\right)\sum_{j=0}^{M-1}\frac{(k^2/4)^j}{j!}$$
$$+ \frac{e^{-|\mathrm{x}|^2}}{\pi^{3/2}}\sum_{j=0}^{M-2}\frac{(-1)^j\, H_{2j+1}(|\mathrm{x}|)}{2^{2j+3}\,|\mathrm{x}|}\sum_{r=0}^{M-2-j}\frac{(k^2/4)^r}{(j+r+1)!} .$$

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